# Selected Problems of Controllability of Semilinear Fractional Systems-A Survey

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**Abstract.** The following article presents recent results of controllability problem of dynamical systems in infinite and finite-dimensional spaces. Roughly speaking, we describe selected controllability problems of fractional order systems, including approximate controllability and complete controllability.

**Keywords:** Fractional systems · Controllability · Fixed point theorem · Banach space

## 1 Introduction

Controllability is very important property of dynamical systems and it plays a crucial role in many control problems. The assumption that the control system is controllable performs fundamental establishment among other in optimal control, stabilizability and pole placement problem [26,35]. In general controllability means that there exists a control function which steers the solution of the dynamical system from its initial state to a final state using a set of admissible controls, where the initial and final states may vary over the entire space. A standard approach to the controllability problems is to transform it into a fixed point problem for an appropriate operator in a functional space. There are many studies are related to the controllability problem. In [5,6,15,20,23,24,28,30] authors used the theory of fractional calculus. A fixed point approach we can find in [3,12,17,37].

Nowadays the fractional calculus and its applications in control theory are very popular and they became standard tool in designing of control systems. Moreover, the fractional calculus has become a powerful tool in modeling several complex phenomena in numerous seemingly diverse and widespread fields such as engineering, chemistry, mechanics, aerodynamics, physics, etc. [13,14].

A lot of dynamical systems based on mathematical modelling of realistic models can be described as partial fractional differential or integrodifferential inclusions [4,9,32,33]. A new approach to obtain the existence of mild solutions


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and controllability results are presented in [41]. For this purpose they avoid hypotheses of compactness on the semigroup generated by the linear part and any conditions on the multivalued nonlinearity expressed in terms of measures of noncompactness. Author of [39] studied fractional evolution equations and inclusions and they presented application of obtained results in control theory. Many authors [7,8,21,31,40] investigated the existence of solutions for fractional semilinear differential or integrodifferential equations.

The special case of dynamical systems so-called the impulsive differential systems can be used to model processes which are subject to sudden changes and which cannot be described by the standard types of differential systems [22]. In [36] authors considered the controllability problem for impulsive differential and integrodifferential systems in Banach spaces. Articles [29] and [38] are devoted to the controllability of fractional evolution systems. The problem of controllability and optimal controls for functional differential systems has been extensively investigated in many articles [1,2].

In this paper we discuss selected problems of controllability for various types of fractional order systems. We present the latest results for finite and infinite-dimensional fractional nonlinear systems.

## 2 Basic Notations

In this section, we introduce some definitions. Let $(X, \|\cdot\|)$ be a Banach space, $J = [0, t_1]$, $\alpha \in (0, 1)$ and $f : J \to X$ be a given function.

**Definition 1** [16]**.** *The Caputo fractional derivative of order $\alpha$ is given as follows*

$$^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f^{'}(s)ds}{(t-s)^{\alpha}},$$

*where: $f$ is the function which has absolutely continuous derivative, $\Gamma$ is the Gamma function and $f^{'}$ is the derivative of function $f$.*

For completeness of presentation, below the definition of the measure noncompactness is shown. It is the generalization of the Schauder's theorem [18].

**Definition 2.** *Let $(X, \|\cdot\|)$ be a Banach space and $E$ be a bounded subset of $X$. Then the measure noncompactness of the set $E$ is defined as*

$$\mu(E) = \inf\{r > 0 : \textit{E can be covered by}$$
$$\textit{a finite number of balls whose radii are smaller than } r\}.$$

**Theorem 1 (Darbo fixed point theorem)** [27]**.** *Let $Q$ be a nonempty, bounded, convex and closed subset of the space $X$ and let $F : Q \to Q$ be a continuous function such that*

$$\mu(F(S)) \leq k\mu(S),$$

*for all nonempty subset $S$ of $Q$, where $k \in [0, 1)$ is a constant. Then $F$ has a fixed point in the set $Q$.*

**Theorem 2 (Schauder's fixed point theorem)** [19]**.** *Let $Q$ be a closed convex subset of $X$ and $F : Q \to Q$ be a continuous function. Then $F$ has at least one fixed point in the set $Q$.*

**Theorem 3 (Krasnoselskii fixed point theorems)** [11]**.** *Let $X$ be a Banach space and $Q$ be a bounded, closed, and convex subset of $X$. Let $V_1$, $V_2$ be maps of $Q$ into $X$ such that $V_1 x + V_2 y \in Q$ for every $x, y \in Q$. If $V_1$ is a contraction and $V_2$ is compact and continuous, then equation $V_1 x + V_2 x = x$ has a solution on $Q$.*

Moreover the relative and approximate controllability definitions are recall below.

**Definition 3** [25]**.** *The dynamical system is said to be relative controllable on interval $[0, t_1]$, if for every initial function $\phi$ and every final state $x_1 \in \mathbb{R}^n$ there exists a control function $u$ defined on $[0, t_1]$ such that the solution of dynamical system satisfies $x(t_1) = x_1$.*

**Definition 4.** *The dynamical system is said to be approximately controllable in time interval $[0, t_1]$, if for every desired final state $x_1$ and $\varepsilon > 0$ there exists a control function $u$ such that the solution of dynamical system satisfies*

$$\|x(t_1) - x_1\| < \varepsilon.$$

## 3 Controllability of Semilinear Fractional Systems

In this section mathematical models of fractional systems with different delays in state and control will be presented. Let us introduce the following necessary notation:

- $0 < \alpha < 1$ is order of derivative,
- $\phi$ is a continuous function on $[-h, 0]$, $h \in [0, \infty)$, $\phi : [-h, 0] \to \mathbb{R}^n$,
- $A, B$ are $n \times n$ dimensional matrices and $C$ is $n \times m$ dimensional matrix,
- $u$ is the control function $u : [-h, \infty) \to \mathbb{R}^m$,
- $\mathcal{L}$ is the Laplace transform,
- $X_\alpha(t) = \mathcal{L}^{-1}\Big[[s^\alpha \cdot I - A - Be^{-s}]^{-1} s^{\alpha-1}\Big](t)$,
- $X_{\alpha,\alpha}(t) = t^{t-\alpha} \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} X_\alpha(s) ds$,
- $x_L(t; \phi) = X_\alpha(t)\phi(0) + \int_{-h}^0 (t - s - h)^{\alpha-1} X_{\alpha,\alpha}(t - s - h)\phi(s) ds$,
- $H(t, s)$ is an $n \times m$ matrix, continuous in $t$ for fixed $s$, $H : J \times [-h, 0] \to \mathbb{R}^{n \times m}$,
- $\int_{-h}^0 d_s H(t, s)$ denotes the integrals in the Lebesgue-Stieltjes sense with respect to $s$.

The next definition will be used in investigation about controllability of semilinear dynamical systems.

**Definition 5** [25]. *Assume that there exist positive real constants $K$ and $k$ with $0 \le k < 1$ such that*

$$|f(t,x,y,z,u)| \le K, \tag{1}$$

$$|f(t,x,y,z,u) - f(t,x,y,\overline{z},u)| \le k(|z-\overline{z}|) \tag{2}$$

*for all $x, y, z, \overline{z} \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, where $f$ is nonlinear function.*

### 3.1 Fractional System with Distributed Delays in Control

The fractional delay dynamical system with distributed delays in control can be presented by the following equation:

$$\begin{gathered} {}^{C}D^{\alpha}x(t) = Ax(t) + Bx(t-h) + \int_{-h}^{0} d_s H(t,s)u(t+s) + \\ + f\big(t, x(t), x(t-h), {}^{C}D^{\alpha}x(t), u(t)\big) \\ x(t) = \phi(t), \quad t \in [-h, 0]. \end{gathered} \tag{3}$$

Using the well-known result of the unsymmetric Fubini theorem [10], the solution of (3) can be expressed by the following form:

$$\begin{aligned} x(t) &= x_L(t;\phi) \\ &+ \int_{-h}^{0} dH_\tau \int_{\tau}^{0} \big(t-(s-\tau)\big)^{\alpha-1} X_{\alpha,\alpha}\big(t-(s-\tau)\big) H(s-\tau,\tau) u_0(s) ds \\ &+ \int_{0}^{t}\int_{-h}^{0} \big(t-(s-\tau)\big)^{\alpha-1} X_{\alpha,\alpha}\big(t-(s-\tau)\big) d_\tau H_t(s-\tau,\tau) u(s) ds \\ &+ \int_{0}^{t} (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) f\big(s, x(s), x(s-h), {}^{C}D^{\alpha}x(s), u(s)\big) ds, \end{aligned}$$

where

$$H_t(s,\tau) = \begin{cases} H(s,\tau), & s \le t, \\ 0, & s > t, \end{cases}.$$

The below, we present the main result of relative controllability for the system (3).

**Theorem 4.** *Assume that the nonlinear function $f$ satisfies the conditions* (1) *and* (2) *and suppose that the controllability Gramian*

$$W = \int_{0}^{t_1} S(t_1,s) S^{*}(t_1,s) ds$$

*where:*

$$S(t,s) = \int_{-h}^{0} \big(t-(s-\tau)\big)^{\alpha-1} X_{\alpha,\alpha}\big(t-(s-\tau)\big) d_\tau H_t(s-\tau,\tau).$$

*is positive definite. Then the nonlinear system* (3) *is relatively controllable on $J$.*

In order to prove of Theorem 4 authors using Darbo fixed point theorem. Specifying the matrix function $H(t,s)$, it is possible to obtain systems with different lumped delays in control.

### 3.2 Fractional Systems with Multiple Delays in Control

In the same article [25] the authors focus on the implicit fractional delay dynamical system with time varying multiple delays in control given by the equation

$$\begin{gathered} {}^C D^\alpha x(t) = Ax(t) + Bx(t-h) + \sum_{i=0}^{M} C_i u(\sigma_i(t)) \\ + f\big(t, x(t), x(t-h), {}^C D^\alpha x(t), u(t)\big), \\ x(t) = \phi(t), \quad t \in [-h, 0], \end{gathered} \tag{4}$$

where $C_i$ for $i = 0, 1, \ldots, M$ are $n \times l$ matrices.

In order to obtain the main result we present some necessary hypotheses.

**Hypothesis 1.** *The functions $\sigma_i : J \to \mathbb{R}$, $i = 0, 1, \ldots, M$, are twice continuously differentiable and strictly increasing in $J$. Moreover $\sigma_i(t) \le t$, $i = 0, 1, \ldots, M$, for $t \in J$.*

**Hypothesis 2.** *Introduce the time lead functions $r_i(t) : [\sigma_i(0), \sigma_i(t_1)] \to [0, t_1]$, $i = 0, 1, \ldots, M$, such that $r_i(\sigma_i(t)) = t$ for $t \in J$. Further, $\sigma_0(t) = t$ and for $t = t_1$ the following inequality holds:*

$$\begin{gathered} \sigma_M(t_1) \le \cdots \le \sigma_{l+1}(t_1) \le 0 = \sigma_l(t_1) \\ < \sigma_{l-1}(t_1) = \cdots = \sigma_1(t_1) = \sigma_0(t_1) = t_1. \end{gathered} \tag{5}$$

**Hypothesis 3.** *Given $\sigma > 0$, for functions $u : [-\sigma, t_1] \to \mathbb{R}^l$ and $t \in t_1$, we use the symbol $u_t$ to denote the function on $[-\sigma, 0]$ defined by $u_t(s) = u(t+s)$ for $s \in [-\sigma, 0)$.*

Using (5), we can write solution of (4):

$$\begin{gathered} x(t) = x_L(t; \phi) + H(t) \\ + \sum_{i=0}^{l} \int_0^t (t - r_i(s))^{\alpha-1} X_{\alpha,\alpha}(t - r_i(s)) C_i \dot{r}_i(s) u(s) ds \\ + \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) f\big(s, x(s), x(s-h), {}^C D^\alpha x(s), u(s)\big) ds, \end{gathered} \tag{6}$$

where

$$\begin{aligned} H(t) = &\sum_{i=0}^{l} \int_{\sigma_i(0)}^{0} (t - r_i(s))^{\alpha-1} X_{\alpha,\alpha}(t - r_i(s)) C_i \dot{r}_i(s) u_0(s) ds \\ &+ \sum_{i=l+1}^{M} \int_{\sigma_i(0)}^{\sigma_i(t)} (t - r_i(s))^{\alpha-1} X_{\alpha,\alpha}(t - r_i(s)) C_i \dot{r}_i(s) u_0(s) ds. \end{aligned}$$

The main theorem is given by the following form.

**Theorem 5.** *Assume that the Hypotheses 1–3 hold. Further assume that the nonlinear function satisfies the condition* (1) *and* (2) *and suppose that determinant of Gramian matrix*

$$W = \sum_{i=0}^{l} \int_0^{t_1} \Big(X_{\alpha,\alpha}(t_1 - r_i(s))C_i\dot{r}_i(s)\Big)\Big(X_{\alpha,\alpha}(t_1 - r_i(s))C_i\dot{r}_i(s)\Big)^T ds$$

*is positive definite. Then the nonlinear system* (4) *is relatively controllable on* $J$.

As before, the proof was obtained using Darbo fixed point theorem.

## 4 The Controllability of Nonlocal Nonlinear Fractional Systems

In this section, we present a recent results concerning nonlinear fractional system in infinite-dimensional space.

### 4.1 Approximate Controllability of Fractional Nonlocal Evolution Equation with Multiple Delays

Authors of [34] consider dynamical system described as follows:

$$\begin{gathered} {}^C D^\alpha x(t) = Ax(t) + f\left(t, x(t), x(t-\tau_1), \ldots, x(t-\tau_n)\right) + Bu(t), \\ t \in J = [0, T], \quad x(t) + g(x) = \varphi(t), \quad t \in [-b, 0], \end{gathered} \tag{7}$$

where ${}^C D^\alpha$ is Caputo fractional derivative of order $\alpha \in (0, 1)$, $T > 0$ is a constant, $A$ generates a compact analytic semigroup $S(t)\,(t \geq 0)$ of uniformly bounded linear operator, $u \in L^2(J, X)$ is a control, $X$ is a Banach space, $X_\alpha$ is the Banach space of $D(A^\alpha)$ with norm $\|x\|_\alpha := \|A^\alpha x\|$ for any $x \in D(A^\alpha)$, $B : X \to X_\alpha$ is a linear bounded operator, $\tau_1, \tau_2, \ldots, \tau_n$ are positive constants, $b = \max\{\tau_1, \tau_2, \ldots, \tau_n\}$, $\varphi : [-b, 0] \to X_\alpha$ is continuous, $f$ and $g$ are given functions.

To obtain the main results it should pose a some assumptions.

**Hypothesis 4.** *The function* $f : J \times X_\alpha^{n+1} \to X$ *is continuous and there exist positive constants* $\beta_0, \beta_1, \ldots, \beta_n$ *and* $K \geq 0$ *such that*

$$\|f(t, \nu_0, \nu_1, \ldots, \nu_n)\| \leq \sum_{i=0}^{n} \beta_i \|\nu_i\|_\alpha + K, \quad t \in J,\ (\nu_0, \nu_1, \ldots, \nu_n) \in X_\alpha^{n+1}.$$

**Hypothesis 5.** *The function* $g : C\left([-b, T], X_\alpha\right) \to X_\alpha$ *is continuous and there exists a constant* $L \geq M$, $M \geq 1$, *such that*

$$\|g(x) - g(y)\|_\alpha \leq \frac{\|x - y\|_C}{L + \|x - y\|_C}, \quad x, y \in C\left([-b, T], X_\alpha\right),$$

*where* $C\left([-b, T], X_\alpha\right)$ *is the Banach space of all continuous* $X_\alpha$*-valued functions on the interval* $[-b, T]$ *with norm* $\|x\|_C = \max_{t \in [-b,T]} \|x(t)\|_\alpha$ *for any* $x \in C\left([-b, T], X_\alpha\right)$.

**Hypothesis 6.** *The function* $f : J \times X_\alpha^{n+1} \to X$ *is bounded.*

**Hypothesis 7.** *The linear fractional system given in the form:*

$$
\begin{aligned}
{}^C D^\alpha x(t) &= Ax(t) + Bu(t), \quad t \in [0, T], \\
x(0) &= x_0 \in X_\alpha
\end{aligned}
$$

*is approximately controllable.*

Then, we can present the main results of [34].

**Theorem 6.** *Assume that the Hypotheses 4–7 hold. Then the fractional nonlocal control system* (7) *is approximately controllable.*

To prove obtained results author used fixed point theory (see [34]).

## 4.2 Approximate Controllability of Impulsive Nonlocal Nonlinear Fractional Systems

The controllability problem for impulsive nonlocal nonlinear fractional systems is discussed in [11]. That system is given in the following form:

$$
\begin{aligned}
{}^C D_\alpha x(t) &= Ax(t) + f\left(t, x(t), (Wx)(t)\right) + Bu(t), \\
& t \in (0, b] \setminus \{t_1, t_2, \dots, t_m\}, \\
x(0) + g(x) &= x_0 \in X,\ \Delta x(t_i) = I_i\left(x(t_i^-)\right) + D\nu(t_i^-), \\
& i = 1, 2, \dots, m,
\end{aligned} \tag{8}
$$

where ${}^C D_\alpha$ is the Caputo fractional derivative of order $\alpha \in (0, 1)$, the state $x(\cdot)$ takes its values in a Banach space $X$ with norm $\|\cdot\|$, and $x_0 \in X$. Let $A : D(A) \subset X \to X$ be a sectorial operator of type $(M, \theta, \alpha, \mu)$ on $X$, $\mu \in \mathbb{R}$, $0 < \theta < \frac{\pi}{2}$, $M > 0$, $W : I \times I \times X \to X$ represents a Volterra-type operator such that $(Wx)(t) = \int_0^t h(t, s, x(s))ds$, the control functions $u(\cdot)$ and $\nu(\cdot)$ are given in $L^2(I, U)$, $U$ is a Banach space, $B$ and $D$ are bounded linear operators from $U$ into $X$. Here, one has $I = [0, b]$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, $I_i : X \to X$ are impulsive functions that characterize the jump of the solutions at impulse points $t_i$, the nonlinear term $f : I \times X \times X \to X$, the nonlocal function $g : PC(I, X) \to X$, where $PC(I, X)$ is the space of $X$-valued bounded functions on $I$ with the uniform norm $\|x\|_{PC} = \sup\{\|x(t)\|, t \in I\}$ such that $x(t_i^+)$ exists for any $i = 0, \dots, m$ and $x(t)$ is continuous on $(t_i, t_{i+1}]$, $t_0 = 0$ and $t_{m+1} = b$, $\Delta x(t_i) = x(t_i^+) - x(t_i^-)$, $x(t_i^+)$ and $x(t_i^-)$ are the right and left limits of $x$ at the point $t_i$, respectively.

The main assumptions are formulated as follows.

**Hypothesis 8.** *The operators* $S_\alpha(t) = \frac{1}{2\pi i}\int_c e^{\lambda t}\lambda^{\alpha-1}R(\lambda^\alpha, A)d\lambda$ *and* $T_\alpha(t) = \frac{1}{2\pi i}\int_c e^{\lambda t}R(\lambda^\alpha, A)d\lambda$ *with* $c$ *being a suitable path such that* $\lambda^\alpha \notin \mu + S_\theta$ *for* $\lambda \in c$, *generated by* $A$, *are bounded and compact, such that* $\sup_{t\in I}\|S_\alpha\| \le M$ *and* $\sup_{t\in I}\|T_\alpha\| \le M$.

**Hypothesis 9.** *The nonlinearity $f : I \times X \times X \to X$ is continuous and compact; there exist functions $\mu_i \in L^\infty(I, \mathbb{R}^+)$, $i = 1, 2, 3$, and positive constants $q_1$ and $q_2$ such that $\|f(t, x, y)\| \le \mu_1(t) + \mu_2(t)\|x\| + \mu_3(t)\|y\|$ and $\|f(t, x, Wx) - f(t, y, Wy)\| = q_1\|x - y\| + q_2\|Wx - Wy\|$.*

**Hypothesis 10.** *Function $g : PC(I, X) \to X$ is completely continuous and there exists a positive constant $\beta$ such that $\|g(x) - g(y)\| \le \beta\|x - y\|$, $x, y \in X$.*

**Hypothesis 11.** *Associated with $h : \Delta \times X \to X$, there exists $m(t, s) \in PC(\Delta, \mathbb{R}^+)$ such that $\|h(t, s, x(s))\| \le m(t, s)\|x\|$ for each $(t, s) \in \Delta$ and $x, y \in X$, where $\Delta = \{(t, s) \in \mathbb{R}^2 | t_i \le s,\ t \le t_{i+1},\ i = 0, \ldots, m\}$.*

**Hypothesis 12.** *For every $x_1, x_2, x \in X$ and $t \in (t_i, t_{i+1}]$, $i = 1, \ldots, m$, $I_i$ are continuous and compact and there exist positive constants $d_i$, $e_i$ such that*

$$\|I_i\left(x_1(t_i^-)\right) - I_i\left(x_2(t_i^-)\right)\| \le d_i \sup_{t \in (t_i, t_{i+1}]} \|x_1(t) - x_2(t)\|$$

*and*

$$\|I_i\left(x(t_i^-)\right)\| \le e_i \sup_{t \in (t_i, t_{i+1}]} \|x(t)\|.$$

Additionally, to formulate the main results it have to define a relevant operator:

$$\mathcal{R}(\lambda, \Psi_{t_{k-1},i}^{t_k}) = \left(\lambda I + \Psi_{t_{k-1},i}^{t_k}\right)^{-1}, \quad i = 1, 2$$

for $\lambda > 0$, where

$$\Psi_{t_{k-1},1}^{t_k} = \int_{t_{k-1}}^{t_k} T_\alpha(t_k - s) B B^* T_\alpha^*(t_k - s) ds, \quad k = 1, 2 \ldots, m + 1,$$
$$\Psi_{t_{k-1},2}^{t_k} = S_\alpha(t_k - t_{k-1}) D D^* S_\alpha^*(t_k - t_{k-1}), \quad k = 2, 3, \ldots, m + 1$$

are the controllability operators associated with the linear impulsive fractional control system:

$$\begin{aligned} {}^C D^\alpha x(t) &= Ax(t) + Bu(t) \\ x(0) &= x_0 \in X, \\ \Delta x(t_i) &= D\nu(t_i^-), \quad i = 1, \ldots, m. \end{aligned} \tag{9}$$

The main theorem of [11] is formulated as follows:

**Theorem 7** [11]. *If Hypotheses 8–12 are satisfied and $\lambda \mathcal{R}\left(\lambda, \Psi_{t_{k-1},i}^{t_k}\right) \to 0$ in the strong operator topology as $\lambda \to 0^+$, $i = 1, 2$, then the impulsive nonlocal fractional system (8) is approximately controllable on $t \in [0, b] \setminus \{t_1, \ldots, t_m\}$.*

For prove presented results authors of [11] used fractional calculus, sectorial operators and Krasnoselskii fixed point theorems.

## 5 Conclusions

After scrutinizing the selected articles in presented survey we observe a research methodology, which is used to solve the controllability problem. Below it is shown the methodology resulting from in-depth analysis of the papers concerning the controllability of semilinear and nonlinear fractional systems:

(a) showing a mathematical model of dynamical system;
(b) formulation the assumptions concerned dynamical systems;
(c) proof of solution existence of state-space equation using various types of fixed-point theorem or generally fixed-point technique;
(d) formulation theorem contains necessary conditions of controllability;
(e) proof of the above-mentioned theorem.

We also notice that the main role plays the assumption about Lipschitz continuity.

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